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Solutions for Steady Plane Orthogonal MHD Flows

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Steady, plane flow of an inviscid, electrically conducting incompressible fluid with infinite electrical conductivity is considered and a single partial differential equation, in the case of orthogonal flow, is obtained which involves two functions. Appropriate specialization of these functions generates new exact solutions of the original equations.

1. INTRODUCTION

In recent years the study of flow problems of electrically conducting fluids has received considerable interest. Such studies have been made for many years in connection with astrophysical and geophysical problems, such as sun spot theory, the motion of the interstellar gas, the origin of the earths' magnetism, etc. Recently, engineering problems involving, e.g., controlled fusion research, reentry problems of intercontinental ballistic missiles, plasma jets, communications, and power conversion have required studies of the flow of an electrically conducting fluid.

A vast amount of research has been carried out on the motion of electrically conducting fluids moving in a magnetic field since Alfven's (1950) classic work. The mathematical complexity of the phenomenon has induced many researchers to adopt a rather useful alternate technique of investigating special classes of flows, such as aligned flows, orthogonal flows, or transverse flows. These special classes of flows yielded various solvable second-order mathematical structures, and these structures aided in the determination of similarities and contrasts with ordinary fluid dynamics. These results were often achieved by employing well-established fluid-dynamic techniques. For example, in the case of an inviscid incompressible fluid in steady flow, Ladikov (1962) derived two Bernoulli-type

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equations for orthogonal flows, Kingston and Talbot (1969) classified all orthogonal flows as radial, vortex, rectilinear, or as certain types of spirals, and Chandna and Nath (1972) established uniqueness properties for aligned flows.

In this article a different approach (Rogers, 1971) is employed for orthogonal flows of an inviscid, electrically conducting incompressible fluid. A single partial differential equation which involves two functions is obtained. Appropriate specialization of these functions generates new exact solutions of the original equations.

2. BASIC EQUATIONS

The equations of motion governing the steady flow of an inviscid, incompressible fluid with infinite conductivity in the presence of a magnetic field are (Singh and Singh, 1985)

$$\operatorname{div} \mathbf{q} = 0 \tag{2.1}$$

$$\rho(\mathbf{q}\cdot\operatorname{grad})\mathbf{q} + \nabla p + \mu \mathbf{H}\mathbf{x} \operatorname{curl} \mathbf{H} = \mathbf{0} \tag{2.2}$$

$$\operatorname{Curl}(\mathbf{q}\times\mathbf{H}) = \mathbf{0} \tag{2.3}$$

$$\operatorname{div} \mathbf{H} = 0 \tag{2.4}$$

where $\mathbf{q}$ is the velocity field vector, $\mathbf{H}$ is the magnetic field vector, p is the fluid pressure, and the constants μ and ρ are the magnetic permeability and fluid density, respectively.

Equation (2.2) can be written as

$$\rho(\mathbf{q}\cdot\operatorname{grad})\mathbf{q} + \nabla p + \tfrac{1}{2}\mu\nabla H^2 = \mu(\mathbf{H}\cdot\operatorname{grad})\mathbf{H} \tag{2.5}$$

where H is the magnitude of the magnetic field $\mathbf{H}$. We now consider the flow to be two dimensional, so that $\mathbf{q}$ and $\mathbf{H}$ lie in a plane defined by the rectangular coordinates (x, y) and all the flow variables are functions of x, y. Therefore the above system of equations is replaced by the system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.6}$$

$$\frac{\partial p^*}{\partial x} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\left(H_1\frac{\partial H_1}{\partial x} + H_2\frac{\partial H_2}{\partial y}\right) \tag{2.7}$$

$$\frac{\partial p^*}{\partial y} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = \mu\left(H_1\frac{\partial H_2}{\partial x} + H_2\frac{\partial H_2}{\partial y}\right) \tag{2.8}$$

$$uH_2 - vH_1 = h \tag{2.9}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \tag{2.10}$$

where (u, v) are the velocity components, (H_1, H_2) the components of the magnetic field vector, $p^* = p + \frac{1}{2}\mu H^2$, and h is an arbitrary constant which is zero for aligned flow and nonzero in the case of nonaligned flows.

Employing equations (2.6) and (2.10), we can replace equations (2.7) and (2.8)

$$\frac{\partial}{\partial x}(p^* + \rho u^2 - \mu H_1^2) + \frac{\partial}{\partial y}(\rho uv - \mu H_1 H_2) = 0 \tag{2.11}$$

$$\frac{\partial}{\partial x}(\rho uv - \mu H_1 H_2) + \frac{\partial}{\partial y}(p^* + \rho v^2 - \mu H_2^2) = 0 \tag{2.12}$$

We now study orthogonal flow. With **H** in the plane of flow, we have

$$uH_1 + vH_2 = 0 \tag{2.13}$$

Equations (2.9) and (2.12) yield

$$H_1 = \frac{-hv}{u^2 + v^2}, \qquad H_2 = \frac{hu}{u^2 + v^2} \tag{2.14}$$

By the use of equation (2.14), equations (2.11) and (2.12) may be written as

$$\frac{\partial}{\partial x}\left[p^* + \rho u^2 - \frac{\mu h^2 v^2}{(u^2+v^2)^2}\right] + \frac{\partial}{\partial y}\left[uv\left(\rho + \frac{\mu h^2}{(u^2+v^2)^2}\right)\right] = 0 \tag{2.15}$$

and

$$\frac{\partial}{\partial x}\left[uv\left(\rho + \frac{\mu h^2}{(u^2+v^2)^2}\right)\right] + \frac{\partial}{\partial y}\left[p^* + \rho v^2 - \frac{\mu h^2 u^2}{(u^2+v^2)^2}\right] = 0 \tag{2.16}$$

The equation of continuity (2.6) implies the existence of a stream function $\psi(x, y)$, such that

$$u = \psi_y, \qquad v = -\psi_x \tag{2.17}$$

From equation (2.16) the function $\eta(x, y)$ is such that

$$uv\left(\rho + \frac{\mu h^2}{(u^2+v^2)^2}\right) = -\eta_{xy} \tag{2.18}$$

and

$$p^* + \rho v^2 - \frac{\mu h^2 u^2}{(u^2+v^2)^2} = \eta_{xx} \tag{2.19}$$

Note that $\eta' = \eta_x$ is a Bateman's (1943–44) lift function. From equation (2.19), we may write

$$p^* = \frac{\eta_{xx}\psi_x + \eta_{xy}\psi_y}{x} - \rho(\psi_x\psi_x + \psi_y\psi_y) \tag{2.20}$$

where $\psi_x \neq 0$, $\psi_y \neq 0$, $\eta_{xy} \neq 0$. With the help of equations (2.17), (2.18), and (2.20), equation (2.15) can be transformed as

$$(\psi_y \eta_{xx} - \psi_x \eta_{xy})_x - (\psi_x \eta_{yy} - \psi_y \eta_{xy})\psi_y - g(y)\psi_x \psi_y = 0 \qquad (2.21)$$

where g is an arbitrary function of y and equation (2.21) is a partial differential equation satisfied by ψ and η.

3. SOLUTIONS

Solutions to equation (2.21) are postulated of the form

$$\psi = x^m A(y) \qquad (3.1)$$

$$\eta = x^n B(y) \qquad (3.2)$$

where m, n are nonzero real constants. Substituting the above in (2.21), we get

$$x^n (mAB'' - nA'B')A^{-1} + mnx^{n-2}[mAB' - (n-1)A'B]A'^{-1}$$
$$+ mg(y) = 0 \qquad (A' \neq 0) \qquad (3.3)$$

This equation shows that the only possibility occurs when $n = 2$ and

$$mAB'' - 2A'B' = 0 \qquad (3.4)$$

$$2mAB' - 2A'B + g(y)A' = 0 \qquad (3.5)$$

since the case $n = 0$ is excluded by virtue of the requirement $\eta_{xy} \neq 0$, while taking x arbitrary merely gives trivially that $p^* = g(y)$ vanishes everywhere.

Equation (3.4) yields the relation

$$B' = aA^{2/m} \qquad (3.6)$$

where a is any constant.

A solution of (3.5) is attempted of the type

$$A' = \varepsilon A^{\delta} \qquad (3.7)$$

where $\delta \neq 0$ and $\varepsilon \neq 0$ are constants. From (3.7) we easily find that

$$A = [\varepsilon(1-\delta)y + k]^{1/(1-\delta)} \qquad (3.8)$$

where k is a constant of integration and $\delta \neq 1$. Hence, from (3.6), we get

$$B = \frac{ma[\varepsilon(1-\delta)y + k]^{2/m(1-\delta)+1}}{\varepsilon[\varepsilon + m(1-\delta)]} + j \qquad (3.9)$$

where j is a constant of integration. Employing the equations (3.1), (3.8), and (3.9) in (2.14), (2.17), and (2.20), we obtain the flow variables

$$H_1 = \frac{hm}{x^{m+1}\varepsilon^2 Y^{(2\delta-1)/(1-\delta)} + m^2 x^{m-1} Y^{1/(1-\delta)}} \tag{3.10}$$

$$H_2 = \frac{h\varepsilon}{\varepsilon^2 x^m Y^{\delta/(1-\delta)} + m^2 x^{m-2} Y^{(2-\delta)/(1-\delta)}} \tag{3.11}$$

$$u = x^m \varepsilon[\varepsilon(1-\delta)y + k]^{\delta/(1-\delta)} \tag{3.12}$$

$$v = -mx^{m+1}[\varepsilon(1-\delta)y + k]^{1/(1-\delta)} \tag{3.13}$$

$$p^* = \frac{2m^2 a Y^{[2+m(1-\delta)]/m(1-\delta)} + 2a\varepsilon^2[2+m(1-\delta)]xY^{[m(\delta-1)+2]/m(1-\delta)}}{\varepsilon[2+m(1-\delta)]m} - \rho[m^2x^{2m-2}Y^{2/(1-\delta)} + \varepsilon^2 x^{2m} Y^{2\delta/(1-\delta)}] + 2j \tag{3.14}$$

where $Y = \varepsilon(1-\delta)y + k$. The pressure p is given by

$$\begin{aligned} p &= p^* - \tfrac{1}{2}\mu(H_1^2 + H_2^2) \\ &= \frac{2m^2 a Y^{[2+m(1-\delta)]/m(1-\delta)} + 2a\varepsilon^2[2+m(1-\delta)]xY^{[m(\delta-1)+2]/m(1-\delta)}}{\varepsilon[2+m(1-\delta)]m} \\ &\quad - \rho[m^2x^{2m-2}Y^{2/(1-\delta)} + \varepsilon^2 x^{2m} Y^{2\delta/(1-\delta)}] \\ &\quad - \frac{\mu h^2}{2}\left[\frac{\varepsilon^2}{(m^{m+1}\varepsilon^2 Y^{(2\delta-1)/(1-\delta)} + m^2 x^{m-1} Y^{1/(1-\delta)})^2}\right. \\ &\quad \left. + \frac{m^2}{(\varepsilon^2 x^m Y^{\delta/(1-\delta)} + m^2 x^{m-2} Y^{(2-\delta)/(1-\delta)})^2}\right] + 2j \end{aligned} \tag{3.15}$$

The streamlines of the flow are the curves

$$x^m[\varepsilon(1-\delta)y + k]^{1/(1-\delta)} = C \quad \text{(constant)} \tag{3.16}$$

This implies that

$$x = C[\varepsilon(1-\delta)y + k]^{-1/(1-\delta)m} \tag{3.17}$$

where different values of C (constant) yield separate streamlines.

In particular, for instance, if

$$m = \frac{2}{\delta - 1}, \qquad \delta \neq 0$$

the streamlines are coaxial parabolas. Finally, the magnetic lines are given by

$$(1-\delta)x^2 - m[\varepsilon(1-\delta)y + k]^2 = C_1 \quad \text{(constant)} \tag{3.18}$$

This represents the equation of a hyperbola whose eccentricity is $[(1+m-\delta)/C_1]^{1/2}$ and center has coordinates

$$(0, -k/\varepsilon(1-\delta))$$

In conclusion, it is observed that the physical requirement $p^*>0$ must be met by suitable adjustment of the available arbitrary constants m, a, ε, δ, and k.

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